PHASE RETRIEVAL IN $\ell_2(\mathbb{R})$

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Abstract. Real phase retrieval is one of the most active areas of research today. Although there are many papers on finite dimensional real phase retrieval, very little work has been done on the infinite dimensional case which is what we will address here. We will review the major results in finite dimensional real phase retrieval for vectors and projections and then see which results extend to infinite dimensions. In particular, we will:

1. show that Edidin’s theorem extends to infinite dimensions.
2. show that there are vectors doing phase retrieval but their perps fail phase retrieval.
3. show there is a family of vectors doing phase retrieval but if any vector is deleted, it fails phase retrieval.
4. extend the notion of full spark to infinite dimensions and see that full spark families do phase retrieval.
5. show that the families of vectors which do phase retrieval are not dense in the families of vectors in $\ell_2$.
6. show that there are families of three Riesz bases that do phase retrieval but leave open the question of where two Riesz bases can do phase retrieval.
7. give several constructions of families of vectors that do phase retrieval in $\ell_2$.

1. INTRODUCTION

Phase retrieval is one of the most applied and studied areas of research today. Phase retrieval for Hilbert space frames was introduced in [2] and quickly became an industry. Although much work has been done on the complex infinite dimensional case of phase retrieval, only one paper exists on infinite dimensional real phase retrieval [3]. Here we will review the major results on finite dimensional real phase retrieval and show:

1. Which results hold in infinite dimensions;
2. Which results fail in infinite dimensions;
3. Which results are unknown in infinite dimensions.

We will need the definition of a Hilbert space frame.
Definition 1.1. A family of vectors \( \{x_i\}_{i \in I} \) in a finite or infinite dimensional Hilbert space \( \mathbb{H} \) is a frame if there are constants \( 0 < A \leq B < \infty \) so that
\[
A\|x\|^2 \leq \sum_{i \in I} |\langle x, x_i \rangle|^2 \leq B\|x\|^2, \quad \text{for all } x \in \mathbb{H}.
\]

1. If \( A = B \) this is an A-tight frame.
2. If \( A = B = 1 \), this is a Parseval frame.

We also need to work with Riesz sequences.

Definition 1.2. A family \( X = \{x_i\}_{i \in I} \) in a finite or infinite dimensional Hilbert space \( \mathbb{H} \) is a Riesz sequence if there are constants \( 0 < A \leq B < \infty \) satisfying for all sequences of scalars \( \{a_i\}_{i \in I} \) we have:
\[
A \sum_{i \in I} |a_i|^2 \leq \| \sum_{i \in I} a_i x_i \| \leq B \sum_{i \in I} |a_i|^2.
\]
If the closed linear span of \( X \) equals \( \mathbb{H} \), we call \( X \) a Riesz basis.

The complement property and full spark will be a major tool here.

Definition 1.3. A family of vectors \( \{x_i\}_{i=1}^{\infty} \) in \( \ell^2 \) has the complement property if for every \( I \subset \mathbb{N} \) either \( \text{span}_{i \in I} = \ell^2 \) or \( \text{span}_{i \in I^c} = \ell^2 \).

Definition 1.4. A family of vectors \( \{x_i\}_{i=1}^{m} \) in \( \mathbb{R}^n \) is full spark if for every \( I \subset [m] \) with \( |I| = n \) we have that \( \{x_i\}_{i \in I} \) is linearly independent (hence spans \( \mathbb{R}^n \)).

Throughout the paper, \( \{e_i\}_{i=1}^{\infty} \) will be used to denote the canonical orthonormal basis for the real Hilbert space \( \ell^2 \).

2. Finite Dimensional Results Which Carry Over to Infinite Dimensions

In this section we look at finite dimensional real phase retrieval and norm retrieval results which carry over to infinite dimensions.

2.1. Phase Retrieval. We start with the definitions.

Definition 2.1. A family of vectors \( \{x_i\}_{i \in I} \) in \( \ell^2 \) does phase retrieval if whenever \( x, y \in \ell^2 \) satisfy
\[
|\langle x, x_i \rangle| = |\langle y, x_i \rangle|, \quad \text{for all } i = 1, 2, \ldots,
\]
then \( x = \pm y \).

A family of projections \( \{P_i\}_{i \in I} \) in \( \ell^2 \) does phase retrieval if whenever \( x, y \in \ell^2 \) and
\[
\|P_i x\| = \|P_i y\|, \quad \text{for all } i = 1, 2, \ldots,
\]
then \( x = \pm y \).

The following result appeared in [3]. The corresponding finite dimensional result first appeared in [4].
Theorem 2.2. A family of vectors in real $\ell^2$ does phase retrieval if and only if it has the complement property.

It follows that results on phase retrieval in the finite dimensional case which just depend on the complement property will also hold in $\ell^2$.

Theorem 2.3. Let $X = \{x_i\}_{i=1}^{\infty}$ do phase retrieval.

1. Then so does $\{a_i x_i\}_{i=1}^{\infty}$ for $a_i \neq 0$ for all $i$.
2. If $T$ is an invertible operator then $\{Tx_i\}_{i=1}^{\infty}$ does phase retrieval.

The finite dimensional version of the next theorem first appeared in [7].

Theorem 2.4. A family of projections $\{P_i\}_{i=1}^{\infty}$ on $\ell^2$ does phase retrieval in $\ell^2$ if and only if for every $0 \neq x \in \ell^2$, span$\{P_i x\}_{i=1}^{\infty} = \ell^2$.

Proof. (⇒) We proceed by way of contradiction. So assume that there is an $0 \neq x \in \ell^2$ and $\{P_i x\}_{i=1}^{\infty}$ does not span $\ell^2$. Choose $0 \neq y \in \ell^2$ so that $y \perp P_i x$ for all $i = 1, 2, \ldots$. Let $u = x + y$ and $v = x - y$. Then since $P_i y \perp P_i x$ for all $i$, we have that

\[ \|P_i (x+y)\|^2 = \|P_i x\|^2 + \|P_i y\|^2 = \|P_i (x-y)\|^2. \]

If $\{P_i\}_{i=1}^{\infty}$ does phase retrieval, then

\[ x + y = \pm (x - y). \]

This implies $x = 0$ or $y = 0$, which is a contradiction.


So results in finite dimensions which depend only on Edidin’s theorem also hold in $\ell^2$.

Corollary 2.5. The following are equivalent for a family of projections $\{P_i\}_{i=1}^{\infty}$ on $\ell^2$:

1. $\{P_i\}_{i=1}^{\infty}$ fails phase retrieval.
2. There are vectors $\|x\| = \|y\| = 1$ in $\ell^2$ so that $P_i x \perp P_i y$ for all $i$.

The finite dimensional version of the following theorem appeared in [4]. The proof in that case immediately works in $\ell^2$.

Theorem 2.6. Let $\{P_i\}_{i=1}^{\infty}$ do phase retrieval on $\ell^2$. Then $\{(I - P_i)\}_{i=1}^{\infty}$ does phase retrieval if and only if it does norm retrieval.

In the finite dimensional case it is known [4] that there are projections $\{P_i\}_{i=1}^{m}$ which do phase retrieval but $\{(I - P_i)\}_{i=1}^{m}$ fails phase retrieval. We now give the infinite dimensional version of this.

Theorem 2.7. There is a set of vectors in $\ell^2$ which do phase retrieval but their perps fail to do phase retrieval.
Proof. First, we will show that for each $n \in \mathbb{N}$, there exists a full spark set of $(2n - 1)$ vectors $\{x_{nk}\}_{k=1}^{2n-1}$ in $\mathbb{R}^n$ such that the first coordinate of the vector $(I - P_{nk})(\varphi_n)$ equals zero, where

$$\varphi_n = \left(1, \frac{1}{2}, \ldots, \frac{1}{n}\right)$$

and $P_{nk}$ is the orthogonal projection onto $x_{nk}$.

Define the following subset of $\mathbb{R}^n$:

$$A_n := \left\{ \left( \sum_{i=1}^{n-2} t^{2i} + \left( n - n \sum_{i=2}^{n-1} \frac{t^{i-1}}{i} \right)^2, t, t^2, \ldots, t^{n-2}, n - n \sum_{i=2}^{n-1} \frac{t^{i-1}}{i} \right) : t \in \mathbb{R} \right\}.$$

Let any $x \in A_n$, and denote by $P_x$ the projection onto $x$. Then

$$(I - P_x)(\varphi_n) = \varphi_n - \langle \varphi_n, \frac{x}{\|x\|} \rangle \frac{x}{\|x\|}.$$

Denote by $a_1$ and $b_1$ the first coordinate of $x$ and $(I - P_x)(\varphi_n)$ respectively. Then we have

$$a_1 = \|x\|^2 - a_1^2,$$

and hence

$$b_1 = 1 - \frac{1}{\|x\|^2} (a_1 + 1)a_1 = 0.$$

Now we will show that for any finite family of hyperplanes in $\mathbb{R}^n$, there exists a point in $A_n$ that does not lie in any of these hyperplanes, and therefore there exists a full spark of $(2n - 1)$ vectors $\{x_{nk}\}_{k=1}^{2n-1} \subset A_n$.

Indeed, let $\{W_{ij}\}_{i=1}^k$ be any finite set of hyperplanes in $\mathbb{R}^n$. Suppose, by way of contradiction, that $B_n \subset \bigcup_{i=1}^k W_i$. Then there exists $j \in \{1, \ldots, k\}$ such that $W_j$ contains infinitely many vectors in $A_n$.

Let $u = (u_1, u_2, \ldots, u_n) \in W_j$, $u \neq 0$. Then we have

$$\langle u, x_t \rangle = 0$$

for infinitely many $x_t \in A_n$.

Hence,

$$u_1 \left( \sum_{i=1}^{n-2} t^{2i} + \left( n - n \sum_{i=2}^{n-1} \frac{t^{i-1}}{i} \right)^2 \right) + \sum_{i=1}^{n-2} u_{i+1} t^i + u_n \left( n - n \sum_{i=2}^{n-1} \frac{t^{i-1}}{i} \right) = 0$$

for infinitely many $t \in \mathbb{R}$.

This implies $u_1 = u_2 = \cdots = u_n = 0$, which is impossible.

Thus, we have shown that for each $n$, there exists a full spark set of $(2n - 1)$ vectors $\{x_{nk}\}_{k=1}^{2n-1}$ in $\mathbb{R}^n$ such that the first coordinate of the vector $(I - P_{nk})(\varphi_n)$ equals zero. Notice that $\{x_{nk}\}_{k=1}^{2n-1}$ does phase retrieval in $\mathbb{R}^n$.

Now, for each $n$, we consider $x_{nk}$ as a vector in $\ell_2$, where its $j$-coordinate is zero when $j > n$. Then the collection of all $x_{nk}$, $n \in \mathbb{N}, k = 1, 2, \ldots, 2n-1,$
do phase retrieval in $\ell_2$. Indeed, suppose that $|\langle x, x_{nk} \rangle| = |\langle y, x_{nk} \rangle|$ for all $n, k$ but $x \neq \pm y$. Then there is a $n_0$ such that

\begin{equation}
(x(1), x(2), \ldots, x(n_0)) \neq \pm(y(1), y(2), \ldots, y(n_0)).
\end{equation}

But then the corresponding $\{x_{nk}\}_{k=1}^{2n_0-1}$ does phase retrieval in $\mathbb{R}^{n_0}$ which implies $\langle x(1), x(2), \ldots, x(n_0) \rangle = \pm(y(1), y(2), \ldots, y(n_0))$, a contradiction.

Finally, we show that $\{x_{nk}\}_{n=1,k=1}^\infty$ fails phase retrieval in $\ell_2$. We will use the same notation $P_{nk}$ for the projection onto $x_{nk} \in \ell_2$, and let $\varphi = \{ \frac{1}{n} \}_{n=1}^\infty \in \ell_2$. Then by our construction, the first coordinate of $(I - P_{nk})(\varphi)$ equals zero for all $k, n$. Therefore, $\text{span}\{(I - P_{nk})(\varphi)\}_{nk} \neq \ell_2$. Therefore $\{x_{nk}\}_{n=1,k=1}^{2n-1}$ fails phase retrieval by Theorem 2.4. 

\[ \square \]

In finite dimensions it is known [2] that any family of vectors doing phase retrieval must contain at least $(2n - 1)$-vectors. It follows that a full spark family of vectors $\{x_i\}_{i=1}^{2n-1}$ does phase retrieval (since it has complement property) but if we delete any vector it fails phase retrieval. Now we give a construction to show that this result holds in infinite dimensions. The following example shows that there is a family of vectors in $\ell_2$ which does phase retrieval but we cannot drop any vector and maintain phase retrieval. This also contains a new construction for frames doing phase retrieval.

First, we need the following lemma:

**Lemma 2.8.** Let $\{e_i\}_{i=1}^\infty$ be the canonical orthonormal basis for $\ell_2$. For any fixed $i$, if $x$ is orthogonal to $e_i + e_j$ for infinitely many $j > i$, then $\langle x, e_i \rangle = \langle x, e_j \rangle = 0$ for all such $j$.

**Proof.** Let $K = \{j : j > i, \langle x, e_i + e_j \rangle = 0\}$, then by assumption, the cardinality of $K$ is infinite.

It is clear that $|\langle x, e_i \rangle| = |\langle x, e_j \rangle|$ for all $j \in K$. Suppose by a contradiction that $|\langle x, e_j \rangle| > 0$ for all $j \in K$. Then we have

\begin{equation}
\|x\|^2 \geq \sum_{j \in K} |\langle x, e_j \rangle|^2 = \infty.
\end{equation}

a contradiction. \[ \square \]

**Example 2.9.** Let the family of vectors $X = \{e_i + e_j\}_{i < j}$. Then $X$ does phase retrieval in $\ell_2$ but we cannot drop any vector of $X$ and maintain phase retrieval.

**Proof.** Let $I$ be any subset of the set $\{(i, j) : i < j\}$, and we can assume that $(1, j) \in I$ for infinitely many $j$. We will show that either $\{e_i + e_j\}_{(i, j) \in I}$ or $\{e_i + e_j\}_{(i, j) \in I^c}$ spans $\ell_2$. Suppose $\{e_i + e_j\}_{(i, j) \in I}$ does not span $\ell_2$. We will show that $\{e_i + e_j\}_{(i, j) \in I^c}$ spans $\ell_2$.

Let any $x = (x(1), x(2), \ldots)$ be such that $\langle x, e_i + e_j \rangle = 0$ for all $(i, j) \in I^c$.

By assumption, there is $y = (y(1), y(2), \ldots), y \neq 0$ and $\langle y, e_i + e_j \rangle = 0$ for all $(i, j) \in I$. Let $s$ be the smallest number such that $y(s) \neq 0$. By Lemma
2.8, \((s,j) \notin I\) for infinitely many \(j > s\). Hence there is \(t > s\) such that \((s,j) \in I^c\) for all \(j \geq t\). Again, by Lemma 2.8, we get
\[
\text{(12)} \quad x(s) = x(j) = 0 \text{ for all } j \geq t.
\]

We will now show that \(x(j) = 0\) for all \(j = 1, 2 \ldots t - 1\). Suppose there is \(1 \leq j < s\) such that \(x(j) \neq 0\). This implies \((j,s) \notin I^c\). Thus \((j,s) \in I\) and hence \(y(j) \neq 0\). But this contradicts the way we chose \(s\). So \(x(j) = 0\) for all \(1 \leq j < s\).

Now let any \(s < j < t\). If \((s,j) \in I^c\), then \(x(j) = 0\). If \((s,j) \in I\), then \(y(j) \neq 0\). Note that by assumption, \((1,j) \in I\) for infinitely many \(j\), and hence by Lemma 2.8, we get that \(y(1) = 0\). Thus, \((1,j) \notin I\). Therefore \((1,j) \in I^c\) and so \(x(j) = x(1) = 0\). This completes the proof that \(\{e_i + e_j\}_{(i,j) \in I^c} \text{ span } \ell_2\).

Now we will show that we cannot drop any vector of \(X\) and maintain phase retrieval.

Fix any \((k,\ell), k < \ell\). Consider \(Y = \{e_i + e_j : i < j, (i,j) \neq (k,\ell)\}\). Let
\[
\text{(13)} \quad x = e_k + e_\ell, \quad y = e_k - e_\ell.
\]

Clearly, \(x \neq \pm y\). For any vector \(e_i + e_j \in Y\), we compute:
\[
\text{(14)} \quad \langle x, e_i + e_j \rangle = \langle e_k, e_i \rangle + \langle e_k, e_j \rangle + \langle e_\ell, e_i \rangle + \langle e_\ell, e_j \rangle.
\]

\[
\text{(15)} \quad \langle y, e_i + e_j \rangle = \langle e_k, e_i \rangle + \langle e_k, e_j \rangle - \langle e_\ell, e_i \rangle - \langle e_\ell, e_j \rangle.
\]

If \(i = k\) then \(j \neq \ell\), \(i < \ell\) and \(k < j\). Thus
\[
\text{(16)} \quad \langle x, e_i + e_j \rangle = \langle y, e_i + e_j \rangle = 1.
\]

If \(j = k\), then \(i < j = k < \ell\). So
\[
\text{(17)} \quad \langle x, e_i + e_j \rangle = \langle y, e_i + e_j \rangle = 1.
\]

Consider the case \(i, j \neq k\). If \(i = \ell\) then \(j \neq \ell\). Hence
\[
\text{(18)} \quad \langle x, e_i + e_j \rangle = 1, \text{ and } \langle y, e_i + e_j \rangle = -1.
\]

If \(i \neq \ell\) and \(j = \ell\) then
\[
\text{(19)} \quad \langle x, e_i + e_j \rangle = 1, \text{ and } \langle y, e_i + e_j \rangle = -1.
\]

Finally, if \(i \neq \ell\) and \(j \neq \ell\) then
\[
\text{(20)} \quad \langle x, e_i + e_j \rangle = \langle y, e_i + e_j \rangle = 0.
\]

Thus, in all cases, we always have that
\[
\text{(21)} \quad |\langle x, e_i + e_j \rangle| = |\langle y, e_i + e_j \rangle|, \text{ for all } e_i + e_j \in Y.
\]

Since \(x \neq \pm y\), \(Y\) cannot do phase retrieval. \(\square\)
2.2. Full Spark.

Remark 2.10. It is known that the full spark families of vectors in $\mathbb{R}^n$ are dense in $\mathbb{R}^n$ in the sense that given any family of vectors $x = \{x_i\}_{i=1}^m$ in $\mathbb{R}^n$ and any $\epsilon > 0$, there is a full spark family of vectors $Y = \{y_i\}_{i=1}^m$ in $\mathbb{R}^n$ so that

\[
(22) \quad d(x, y)^2 = \sum_{i=1}^m \|x_i - y_i\|^2 < \epsilon.
\]

One interpretation of the definition of full spark is that any minimal number of vectors in the set which could possibly span, must span, i.e. any subset of $n$-vectors must span. The corresponding statement for $\ell_2$ is:

Definition 2.11. A family of vectors $\{x_i\}_{i=1}^\infty$ is full spark in $\ell_2$ if every infinite subset spans $\ell_2$.

A full spark set clearly has complement property and hence does phase retrieval in the infinite dimensional case.

Theorem 2.12. There exist full spark families of vectors in $\ell_2$ which then do phase retrieval.

Proof. Such an example can be found in Theorem 2 of [10].

There is another simpler way to do this using the following argument. Instead of $\ell_2$ consider $L_2[0,1]$. It is known that if a sequence $a_n \neq a$ of numbers (real or complex) tends to $a$ when $n \to \infty$, then the sequence of functions $f_n(t) = e^{i a_n t}$ spans $L_2[0,1]$ (this is a standard application of the Hahn-Banach theorem together with the uniqueness theorem for holomorphic functions, see more in Appendix III of [8].) Since every subsequence of $a_n$ also has the same limit, every subsequence of $f_n$ also spans $L_2[0,1]$. □

2.3. Norm Retrieval.

Definition 2.13. A family of projections $\{P_i\}_{i \in I}$ (I is finite or infinite) does norm retrieval in $H$ if for any $x, y \in H$ we have:

\[
(23) \quad \|P_i x\| = \|P_i y\| \quad \text{for all } i \in I \text{ then } \|x\| = \|y\|.
\]

The finite dimensional version of the next theorem first appeared in [5].

Theorem 2.14. A family of projections $\{P_i\}_{i=1}^\infty$ on $\ell_2$ does norm retrieval if and only if for any $x \in \ell_2$, $x \in \text{span} \{P_i x\}_{i=1}^\infty$.

Proof. ($\Rightarrow$) Let any $x \in \ell_2$ and $y \perp \text{span} \{P_i x\}_{i=1}^\infty$. Then

\[
(24) \quad \langle P_i x, P_i y \rangle = \langle P_i x, y \rangle = 0, \quad \text{for all } i.
\]

Let $u = x + y, v = x - y$ then we have $\|P_i u\| = \|P_i v\|$ for all $i$. Hence $\|u\| = \|v\|$. Note that $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(u - v)$. Now we compute:

\[
(25) \quad \langle x, y \rangle = \frac{1}{4} \langle u + v, u - v \rangle = \frac{1}{4} (\|u\|^2 - \|v\|^2) = 0.
\]
Thus, \( \overline{\text{span}}\{P_i x\}_{i=1}^{\infty} \perp x \), or equivalently, \( x \notin \overline{\text{span}}\{P_i x\}_{i=1}^{\infty} \).

\( \left(\Rightarrow\right) \) Suppose \( x, y \in \ell_2 \), and \( \|P_i x\| = \|P_i y\| \) for all \( i = 1, 2, \ldots \). Set

\[ u = x + y, v = x - y. \]

Then we have \( \langle P_i u, P_i v \rangle = 0 \) for all \( i \). Hence \( u \perp \overline{\text{span}}\{P_i v\}_{i=1}^{\infty} \). Since \( v \in \overline{\text{span}}\{P_i v\}_{i=1}^{\infty} \) then \( u \perp v \). It follows that \( \|x\| = \|y\| \). \( \square \)

The finite dimensional version of the next result first appeared in [6].

**Theorem 2.15.** A family of vectors \( \{x_i\}_{i=1}^{\infty} \) does norm retrieval in \( \ell_2 \) if and only if for every \( I \subset \mathbb{N} \) if \( x \perp \overline{\text{span}}\{x_i\}_{i \in I} \) and \( y \perp \overline{\text{span}}\{x_i\}_{i \in I^c} \) then \( x \perp y \).

**Proof.** \( \left(\Rightarrow\right) \) We may assume that \( \|x\| = \|y\| = 1 \). Let \( u = x + y, v = x - y \).

Then

\[ |\langle u, x_i \rangle| = |\langle v, x_i \rangle|, \quad \text{for all } i. \]

Since \( \{x_i\}_{i=1}^{\infty} \) does norm retrieval, \( \|u\| = \|v\| \). It follows that \( x \perp y \).

\( \left(\Leftarrow\right) \) Suppose \( |\langle x, x_i \rangle| = |\langle y, x_i \rangle| \), for all \( i \). Denote

\[ I = \{i : \langle x, x_i \rangle = -\langle y, x_i \rangle\}. \]

Then

\[ I^c = \{i : \langle x, x_i \rangle = \langle y, x_i \rangle\}. \]

Let \( u = x + y, v = x - y \). Then \( u \perp \overline{\text{span}}\{x_i\}_{i \in I} \) and \( v \perp \overline{\text{span}}\{x_i\}_{i \in I^c} \).

Therefore, \( u \perp v \), and we get \( \|x\| = \|y\| \). \( \square \)

**Corollary 2.16.** If \( \{x_i\}_{i=1}^{\infty} \) is a Parseval frame in \( \ell_2 \), let for \( I \subset \mathbb{N} \), let

\[ H_1 = \overline{\text{span}}\{x_i\}_{i \in I} \quad \text{and} \quad H_2 = \overline{\text{span}}\{x_i\}_{i \in I^c}, \]

then \( H_1 \perp H_2 \).

### 3. Finite Dimensional Results Which Fail in Infinite Dimensions

It is known [4] that the families of vectors \( \{x_i\}_{i=1}^{m} \) which do phase retrieval in \( \mathbb{R}^n \) are dense in the family of \( m \geq (2n-1) \)-element sets of vectors in \( \mathbb{R}^n \). This follows from the fact that full spark families of \( m \geq 2n-1 \) vectors are dense and do phase retrieval. The corresponding result fails in infinite dimensions.

**Definition 3.1.** We say a family of sequences of vectors \( \mathcal{F} \) is **dense** in \( \ell_2 \) if given any sequence of vectors \( Y = \{y_i\}_{i=1}^{\infty} \subset \ell_2 \) and any \( \epsilon > 0 \) there an \( X = \{x_i\}_{i=1}^{\infty} \in \mathcal{F} \) so that

\[ d(X, Y)^2 = \sum_{i=1}^{\infty} \|x_i - y_i\|^2 < \epsilon. \]

**Remark 3.2.** Note that a Riesz basis cannot do phase retrieval since it clearly fails complement property.
Proposition 3.3. Let \( X = \{x_i\}_{i=1}^{\infty} \subset \ell_2 \) be such that

\[
\sum_{i=1}^{\infty} \|x_i - e_i\|^2 \leq 1 - \epsilon.
\]

Then \( X \) is a Riesz basis for \( \ell_2 \).

Proof. Define an operator \( T : \ell_2 \rightarrow \ell_2 \) by \( Te_i = x_i \), for all \( i = 1, 2, \ldots \). Given \( a = \sum_{i=1}^{\infty} a_i e_i \in \ell_2 \) we have

\[
\|(I - T)a\|^2 = \sum_{i=1}^{\infty} a_i^2 \|x_i - e_i\|^2
\]

\[
\leq \sum_{i=1}^{\infty} |a_i| \|e_i - x_i\|
\]

\[
\leq \left( \sum_{i=1}^{\infty} |a_i|^2 \right) \left( \sum_{i=1}^{\infty} \|x_i - e_i\|^2 \right)
\]

\[
\leq (1 - \epsilon) \|a\|^2.
\]

It follows that \( T \) is an invertible operator and so \( \{Te_i\}_{i=1}^{\infty} \) is a Riesz basis for \( \ell_2 \). \( \square \)

Proposition 3.4. The families of vectors which do phase retrieval in \( \ell_2 \) are not dense in the infinite families of vectors in \( \ell_2 \).

Proof. Let \( 0 < \epsilon < 1 \). If \( X = \{x_i\}_{i=1}^{\infty} \) is any family of unit vectors with

\[
\sum_{i=1}^{\infty} \|x_i - y_i\|^2 < 1 - \epsilon,
\]

then \( X \) is a Riesz basis and hence cannot do phase retrieval. \( \square \)

It is known in finite dimensions [1, 4] that if \( X = \{x_i\}_{i=1}^{m} \) does phase retrieval, there is an \( \epsilon > 0 \) so that whenever \( Y = \{y_i\}_{i=1}^{m} \) satisfies:

\[
\sum_{i=1}^{m} \|x_i - y_i\|^2 < \epsilon,
\]

then \( Y \) does phase retrieval. The above is called a \( \epsilon \)-perturbation of \( X \). The corresponding result fails in \( \ell_2 \) as was shown in [3].

Theorem 3.5. Given a frame \( \{x_i\}_{i=1}^{\infty} \) doing phase retrieval in \( \ell_2 \) and an \( \epsilon > 0 \), there is a frame \( \{y_i\}_{i=1}^{\infty} \) which fails phase retrieval in \( \ell_2 \) and satisfies:

\[
\sum_{i=1}^{\infty} \|x_i - y_i\|^2 < \epsilon.
\]

Definition 3.6. A set of vectors \( \{x_i\}_{i=1}^{\infty} \) in \( \ell_2 \) is finitely full spark if for every \( I \subset \mathbb{N} \) with \( |I| = n \), \( \{Pi x_i\}_{i=1}^{\infty} \) is full spark (i.e. spark \( n+1 \)), where \( P_I \) is the orthogonal projection onto \( \text{span}\{e_i\}_{i \in I} \).
We will have to generalize the definition of full spark.

**Definition 3.7.** A set of vectors \( \{x_i\}_{i=1}^m \) in \( \mathbb{R}^n \) is full spark if either they are independent or if \( m \geq n + 1 \), then they have spark \( n + 1 \).

**Proposition 3.8.** The finitely full spark families of vectors in \( \ell_2 \) are dense in the infinite families of vectors in \( \ell_2 \). In particular, there are Riesz bases for \( \ell_2 \) which are finitely full spark, and these families cannot do phase retrieval.

**Proof.** Let \( \{y_i\}_{i=1}^\infty \) be a family of vectors in \( \ell_2 \) and fix \( \epsilon > 0 \). We will construct the vectors by induction. To get started, choose a vector \( x_1 \) with all non-zero coordinates so that \( \|x_1 - y_1\|^2 < \frac{\epsilon}{2} \). Now assume we have constructed vectors \( \{x_i\}_{i=1}^m \) so that for every \( I \subset \mathbb{N} \) with \( |I| < \infty \), \( \{P_I x_i\}_{i=1}^m \) is full spark and \( \|x_i - y_i\|^2 < \frac{\epsilon}{2^{m+1}} \). For each finite subset \( I \subset \mathbb{N} \), let

\[
G_I = \bigcup \left\{ \text{span} \left[ \{P_{I'} x_i\}_{i \in I'} \cup \{e_{i}\}_{i \in I^c} \right] : I' \subset [m], \begin{cases} |I'| = m & \text{if } m + 1 \leq |I| \\ |I'| = |I| - 1 & \text{if } |I| \leq m \end{cases} \right\}.
\]

Let

\[
F = \bigcup_{n=1}^\infty \bigcup_{|I|=n} G_I,
\]

then \( F \) is a countable union of proper subspaces of \( \ell_2 \) and hence there exists a vector \( y_{m+1} \) not in \( F \) and \( \|x_{m+1} - y_{m+1}\|^2 < \frac{\epsilon}{2^{m+1}} \). This provides the required family of finitely full spark vectors. \( \square \)

## 4. Lifting

In this section we demonstrate an embedding of finite frames in higher dimensions such that the complement property is preserved, which we will refer to as “lifting”. We provide necessary and sufficient conditions for when such a construction is possible and an example to demonstrate problems that may arise in infinite dimensions. We begin with a few useful definitions.

**Definition 4.1.** A frame \( X = \{x_i\}_{i \in I} \) has the overcomplete complement property if for every \( S \subset I \), either \( \{x_i\}_{i \in S} \) or \( \{x_i\}_{i \in S^c} \) spans and is linearly dependent, i.e. it is not a basis.

The overcomplete complement property is a natural generalization of the usual complement property, as will be shown shortly. Next we specify exactly what types of embeddings we are considering.

**Definition 4.2.** A frame \( Y = \{y_i\}_{i=1}^m \subset \mathbb{R}^{n+k} \) is a \( k \)-lifting of a frame \( \{x_i\}_{i=1}^m \) if

\[
y_i|_{\mathbb{R}^n} = x_i, \text{ for all } i = 1, 2, \ldots, m.
\]

The next theorem classifies when 1-lifts are possible and provides a construction for the choice of coordinates to adjoin.
Theorem 4.3. A phase retrievable frame \( X = \{x_i\}_{i=1}^m \subset \mathbb{R}^n \) can be 1-lifted to a phase retrievable frame if and only if \( X \) has the overcomplete complement property.

Proof. For the sufficiency we shall provide a constructive proof. The idea of the proof will be to produce a vector \( v \in \mathbb{R}^m \) such that the \( i \)th coordinate of \( v \) will be the \((n+1)\)th coordinate of \( \hat{x}_i \). Given a subset \( S \subset [m] \), by assumption either \( X_S = \{x_i\}_{i \in S} \) or \( X_{Sc} = \{x_i\}_{i \in Sc} \) spans \( \mathbb{R}^n \) and is linearly dependent. We begin by demonstrating an embedding of vectors from the spanning set that still span in \( \mathbb{R}^{n+1} \). Without loss of generality, in our notation we shall assume \( X_S \) is always the overcomplete spanning set of vectors. Then for some choice of coefficients we have \( \sum_{i \in S} \alpha_i x_i = 0 \) where \( \alpha_i \) are not all zero. Denote \( \alpha_S = (\alpha_1, \alpha_2, \ldots, \alpha_{|S|}) \in \mathbb{R}^{|S|} \) and pick \( \beta_S \in \mathbb{R}^{|S|} \) such that \( \langle \alpha_S, \beta_S \rangle \neq 0 \). Define the embedded vectors \( \hat{X}_S = \{\hat{x}_i\}_{i \in S} \subset \mathbb{R}^{n+1} \) as follows

\[
(42) \quad \hat{x}_i(j) = \begin{cases} 
  x_i(j) & j \in [n] \\
  \beta_S(i) & j = n+1.
\end{cases}
\]

To show that \( \hat{X}_S \) spans \( \mathbb{R}^{n+1} \), observe \( \frac{1}{\langle \alpha_S, \beta_S \rangle} \sum_{i \in S} \alpha_i \hat{x}_i = e_{n+1} \). Since \( X_S \) spans \( \mathbb{R}^n \), it follows that \( \hat{X}_S \) spans \( \mathbb{R}^{n+1} \).

This construction gives a procedure for an embedding which spans the larger space \( \mathbb{R}^{n+1} \), but is dependent on the subset \( S \). Also observe we haven’t posed any conditions on how to extend the vectors in \( S^c \). For each choice of \( S \), we have the associated vectors \( \alpha_S, \beta_S \in \mathbb{R}^{|S|} \). Let \( H_S \subset \mathbb{R}^{|S|} \) denote the hyperplane perpendicular to \( \alpha_S \). Then our construction depends on being able to choose a vector in the complement of \( H_S \) for all subsets \( S \). But the cardinality of \( S \) is changing as we range over all possibilities. To overcome this we will work with the larger space \( \mathbb{R}^m = \mathbb{R}^{|S|} \times \mathbb{R}^{|S^c|} \). There are finitely many choices of \( S \) therefore \( \bigcup_{S \subset [m]} H_S \times \mathbb{R}^{|S^c|} \neq \mathbb{R}^m \). Then for \( v \in \left( \bigcup_{S \subset [m]} H_S \times \mathbb{R}^{|S^c|} \right)^c \), we defined

\[
(43) \quad \hat{x}_i(j) = \begin{cases} 
  x_i(j) & j \in [n] \\
  v(i) & j = n+1.
\end{cases}
\]

Then it follows that \( \hat{X} = \{\hat{x}_i\}_{i=1}^m \) has the complement property in \( \mathbb{R}^{n+1} \).

For necessity assume \( X \) does phase retrieval but does not have the overcomplete complement property. Any spanning set that is a basis cannot be 1-lifted since there will not be enough vectors to span \( \mathbb{R}^{n+1} \).

The result above may be generalized for a \( k \)-lift with minimal effort. Naturally the overcompleteness of each subset \( S \) is critical in determining what integers \( k \) are plausible. More specifically, we define the lifting number of a phase retrievable frame as follows:
Definition 4.4. Given a frame $X = \{x_i\}_{i \in [m]} \subset \mathbb{R}^n$, let
\[
L_X = \min\{|S| - n : \text{span}\{x_i\}_{i \in S} = \mathbb{R}^n \text{ and } |S| \geq |S^c|\},
\]
then $L_X$ is the lifting number for the frame $X$.

From the previous theorem we see immediately that if $\Phi$ has the over-complete complement property then $L_X \geq 1$. The lifting number tells us how many dimensions higher we can lift $\Phi$. If $L_X \geq 1$ then when we 1-lift, each overcomplete spanning subset will be lifted to a spanning set in $\mathbb{R}^{n+1}$ with the same cardinality. If $L_X > 1$ that means each spanning subset with higher cardinality ($S$ or $S^c$) will be lifted to a spanning set which is still not a basis in $\mathbb{R}^{n+1}$, hence can be lifted again. The idea is that after each lift, the lifting number of the subsequent lifted frame $\hat{\Phi}$ is one smaller than $L_X$. That is, if $\hat{X}$ is a 1-lift of $X$ then $L_{\hat{X}} = L_X - 1$.

Corollary 4.5. $X \subset \mathbb{R}^n$ can be $k$-lifted if and only if $k \leq L_X$.

Theorem 4.6. If a frame $X \subset \mathbb{R}^n$ contains $2n + 2m + 1$ vectors with a $2n + 2m$ full spark subset. $X$ can be $(m + 1)$-lifted.

Proof. Clearly if a frame contains a $2n + 2m$ full spark subset it does phase retrieval as it contains a $2n - 1$ full spark subset which already does phase retrieval. Let $X = \{x_i\}_{i \in [2n+2m+1]}$ and $H = \{x_i\}_{i \in [2n+2m]}$ be a full spark subset. Given any $S \subset [2n+2m]$, if $S$ contains more than half the elements in $H$ then it will be a spanning set with more than $n + m$ vectors hence its cardinality minus $n$ will be greater than $m$ hence at least $m + 1$. If it contains less than half of the elements of $H$ then the same holds for $S^c$. If it contained exactly half then both $S$ and $S^c$ will contain $n + m$ elements of $H$ hence they both span. Whichever set that contains $x_{2n+2m+1}$ will be a spanning set of cardinality $n + m + 1$. Hence $X$ will have lifting number $m + 1$.

The set of $2n + 2m + 1$ full spark vectors is open, dense, and contains a subset of $2n + 2m$ full spark vectors. Then the previous theorem shows the set of $2n + 2m + 1$ vectors in $\mathbb{R}^n$ that can be $m + 1$-lifted contains an open dense set. Hence “almost every” set of $2n + 2m + 1$ or $2n + 2m + 2$ vectors can be $m + 1$-lifted.

The infinite dimensional version of this looks like the following. Note that this is not a classification of the liftable phase retrieving frames but a sufficient condition.

Remark 4.7. In $\ell_2$, by a lift we “add” a coordinate at the beginning of the vector. That is, if $\hat{x}$ is a lift of $x$ then $P\hat{x} = (0, x(1), x(2), \ldots)$, where $P$ is the orthogonal projection onto $e_1^\perp$.

Theorem 4.8. Let $X = \{x_i\}_{i=1}^\infty$ be a frame for $\ell_2$ doing phase retrieval and let $Y = \{y_i\}_{i=1}^\infty$ be a linearly dependent spanning set in $\ell_2$. Then $X \cup Y$ can be lifted to a phase retrieving frame for $\ell_2$. 

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Proof. Let $X = \{x_i\}_{i=1}^{\infty}$ and $Y = \{y_i\}_{i=1}^{\infty}$ be as in the theorem. We show that we can lift this union to one higher dimension and maintain phase retrieval. Let $L$ be the right shift operator on $\ell_2$, i.e. if $x = (x(1), x(2), \ldots)$, then $Lx = (0, x(1), x(2), \ldots)$. Replace vectors in $X$ by $\hat{X} = \{\hat{x}_i\}_{i=1}^{\infty}$ where $\hat{x}_i = Lx_i$. The idea for $\hat{Y} = \{\hat{y}_i\}_{i=1}^{\infty}$ is very similar to the proof in Theorem 4.3. We show existence of a vector $v = (v(1), v(2), \ldots) \in \ell_2$ such that $\hat{y}_i = v(i)e_1 + Ly_i$ will have the desired property to assure $\hat{X} \cup \hat{Y}$ does phase retrieval.

Since $\{y_i\}_{i=1}^{\infty}$ is linearly dependent, there exists a sequence of scalars $\alpha = \{\alpha_i\}_{i=1}^{\infty}$ with all but a finite number equal to zero, such that $\sum_{i=1}^{\infty} \alpha_i y_i = 0$. Denote $H_i = e_i^\perp \subset \ell_2$ and note that by the Baire Category Theorem

$$\left((\bigcup_{i=1}^{\infty} H_i) \cup \alpha^\perp\right)^c \neq \emptyset.$$  

Let $v \in \left((\bigcup_{i=1}^{\infty} H_i) \cup \alpha^\perp\right)^c$ and define $\hat{y}_i$ as stated above. Note that $v$ has all non-zero coordinates and $\langle v, \alpha \rangle \neq 0$. Moreover,

$$\sum_{i=1}^{\infty} \alpha_i \hat{y}_i = \sum_{i=1}^{\infty} (\alpha_i(v(i)e_1 + Ly_i)) = \langle \alpha, v \rangle e_1.$$

Let any $j \geq 1$ and $\epsilon > 0$. Since $\{y_i\}_{i=1}^{\infty}$ spans $\ell_2$, there is a finite subset $I_j$ and scalars $\{\beta_k\}_{k \in I_j}$ such that

$$\|e_j - \sum_{k \in I_j} \beta_k y_k\| < \epsilon.$$ 

This implies

$$\|e_{j+1} - \sum_{k \in I_j} \beta_k (\hat{y}_k - v(k)e_1)\| < \epsilon, \text{ for all } j \geq 1.$$

Since $e_1 \in \text{span}\{\hat{y}_i\}_{i=1}^{\infty}$, $e_j \in \text{span}\{\hat{y}_i\}_{i=1}^{\infty}$ for all $j$, $\hat{Y} = \{\hat{y}_i\}_{i=1}^{\infty}$ spans $\ell_2$.

Now we will show that $\hat{X} \cup \hat{Y}$ satisfies Edidin’s theorem. Since $\langle e_1, \hat{y}_i \rangle = v(i) \neq 0$, the projection of $e_1$ on the vectors of $\hat{Y}$ span $\ell_2$. Let any non-zero vector $x \neq e_1$, the projection of $x$ onto the $\hat{x}_i$’s will span $e_1^\perp \subset \ell_2$. Note that $x$ cannot be orthogonal to all $\hat{y}_i$ since these vectors span $\ell_2$. Let $\hat{y}_j$ be one such vector. Since $\hat{y}_j$ is outside of $e_1^\perp \subset \ell_2$, the projection of $x$ onto the vectors of $\hat{X} \cup \hat{Y}$ span $\ell_2$ as well. Hence $\hat{X} \cup \hat{Y}$ does phase retrieval. \qed

5. Finite Dimensional Results which are not Known in Infinite Dimensions

It is known [4] that there are two orthonormal bases for $\mathbb{R}^n$ which do phase retrieval. We do not know the same for $\ell_2$.

Problem 5.1.
Are there two Riesz bases for $\ell_2$ which do phase retrieval?

However, we can do phase retrieval with three Riesz sequences for $\ell_2$. 

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There is a set of vectors

\[ \text{Theorem 6.2.} \]

that for each \( n \) let \( H_n = \text{span}\{e_i\}^{3n+1}_{i=1} \). Choose \( \{u_{nij}\}^{3n+1}_{i=1,j=1} \) so that they are full spark in \( H_n \) and

\[
\sum_{i=3^{n+1}}^{3n+1} \|u_{nij} - e_i\|^2 \leq \frac{1}{2^{n+1}} \quad \text{for } j = 1, 2, 3
\]

Then the collection \( \{u_{nij} : 3^n + 1 \leq i \leq 3^{n+1}, 1 \leq j \leq 3, 1 \leq n < \infty\} \) does phase retrieval. Moreover,

\[
\sum_{n=1}^{\infty} \sum_{i=3^{n+1}}^{3n+1} \|u_{nij} - e_i\|^2 \leq \frac{1}{2} \quad \text{for } j = 1, 2, 3,
\]

therefore \( \{u_{nij}\}^{\infty}_{n=1,i=3^{n+1}} \) is a Riesz sequence for \( j = 1, 2, 3 \). \( \square \)

6. Sets Which Do Phase Retrieval in \( \ell_2 \)

**Theorem 6.1.** Assume we have subspaces \( W_1 \subset W_2 \subset \cdots \subset \ell_2 \) and vectors \( \{x_{ij}\}_{j \in I_i} \) doing phase retrieval in \( W_i \) for every \( i \). Finally, assume \( \cup_{i=1}^{\infty} W_i \) is dense in \( \ell_2 \). Then \( \{x_{ij}\}^{\infty}_{i=1,j \in I_i} \) does phase retrieval in \( \ell_2 \).

**Proof.** We will check the complement property. Observe that a partition of vectors \( \{x_{ij}\}^{\infty}_{i=1,j \in I_i} \) induces a partition for vectors \( \{x_{ij}\}_{j \in I_i} \subset W_i \). By assumption \( \{x_{ij}\}_{j \in I_i} \) does phase retrieval on \( W_i \), therefore for each \( i = 1, 2, \ldots \)

\[
(51) \quad \text{either } W_i \subset \text{span}\{x_{ij}\}_{(i,j) \in I} \text{ or } W_i \subset \text{span}\{x_{ij}\}_{(i,j) \in I^c}.
\]

Then either \( I \) or \( I^c \) contains infinitely many \( W_i \), without loss of generality we assume it is \( I \). This means that for infinitely many \( i \),

\[
(52) \quad W_i \subset \text{span}\{x_{ij}\}_{(i,j) \in I}.
\]

Since \( W_i \subset W_{i+1} \) for all \( i \),

\[
(53) \quad \cup_{i=1}^{\infty} W_i \subset \text{span}\{x_{ij}\}_{(i,j) \in I},
\]

and so the closure of the right hand set is \( \ell_2 \). This shows our family of vectors have complement property and hence do phase retrieval on \( \ell_2 \). \( \square \)

**Theorem 6.2.** Let \( P_n \) be the orthogonal projection of \( \ell_2 \) onto \( E_n = \text{span}\{e_i\}^{n}_{i=1} \). There is a set of vectors \( Y = \{y_{ni}\}^{\infty}_{n=1,i=1} \) that does not do phase retrieval on \( \ell_2 \), but \( X = \{x_{ni}\}^{\infty}_{n=1,i=1} = \{P_n y_{ni}\}^{\infty}_{n=1,i=1} \) does phase retrieval in \( \ell_2 \). Moreover, finite subsets of \( X \) do phase retrieval on \( E_n \) for every \( n \).

**Proof.** For each \( n \in \mathbb{N} \), let \( X_n \) be a finite set of vectors \( \{x_{ni}\}_{i \in I_n} \) contained in \( E_n \) that does phase retrieval in \( E_n \). For example consider a full spark set in \( E_n \) embedded in \( \ell_2 \) by adding zero to all other entries. We know that \( X = \{x_{ni}\}^{\infty}_{n=1,i \in I_n} \) does phase retrieval in \( \ell_2 \). It is sufficient to show that for each \( n \) and \( i \), there exists \( y_{ni} \), with \( P_n y_{ni} = x_{ni} \), such that the \( y_{ni} \)
is contained in a fixed hyperplane for all \( n,i \). Let \( w \) be the vector with infinitely many non-zero coordinates. For each \( n \), \( x_{ni} \) has finite support contained in the first \( n \) coordinates, for all \( i \in I_n \). Then there is \( j > n \) such that \( w(j) \neq 0 \). Define \( y_{ni} = x_{ni} - \left( \frac{\langle x_{ni}, w \rangle}{w(j)} \right) e_j \), for \( i \in I_n \). It follows that \( \langle y_{ni}, w \rangle = 0 \), and hence \( y_{ni} \subset w^\perp \) for all \( n,i \). This completes the proof. \( \square \)

In the following, we will show how to create a new phase retrieval set by translating the vectors of the original one in the same direction. First, we will need a lemma.

**Lemma 6.3.** If \( \{x_i\}_{i=1}^\infty \) is Bessel in \( \ell_2 \), then for every \( v \in \ell_2 \),
\[
\lim_{i \to \infty} \langle v, x_i \rangle = 0.
\]

**Proof.** Given a vector \( v \), we have
\[
\sum_{i=1}^\infty |\langle v, x_i \rangle|^2 < \infty,
\]

hence \( \lim_{i \to \infty} |\langle v, x_i \rangle| = 0. \)

**Remark 6.4.** Note that if any \( \{x_i\}_{i=1}^\infty \) does phase retrieval, then
\[
\left\{ \frac{1}{\|x_i\|} x_i \right\}_{i=1}^\infty
\]
is Bessel and also does phase retrieval.

**Theorem 6.5.** Assume \( \{x_i\}_{i=1}^\infty \) is a Bessel sequence in \( \ell_2 \) and does phase retrieval. Then for every \( v \in \ell_2 \), \( \{x_i + v\}_{i=1}^\infty \) does phase retrieval.

**Proof.** Assume
\[
\langle x, x_i + v \rangle = \langle y, x_i + v \rangle, \text{ for all } i = 1, 2, \ldots.
\]

Let
\[
I = \{ i : \langle x, x_i + v \rangle = \langle y, x_i + v \rangle \}.
\]
Then either \( |I| \) or \( |I^c| \) is infinite. By the complement property, either \( \{x_i\}_{i \in I} \) or \( \{x_i\}_{i \in I^c} \) spans the space. Without loss of generality, assume \( \{x_i\}_{i \in I} \) spans \( \ell_2 \). Now,
\[
\langle x, x_i + v \rangle = \langle y, x_i + v \rangle, \text{ for all } i \in I,
\]

and so
\[
\langle x - y, x_i \rangle = \langle y - x, v \rangle, \text{ for all } i \in I.
\]

By Lemma 6.3,
\[
\langle y - x, v \rangle = 0 = \langle x - y, x_i \rangle, \text{ for all } i \in I.
\]

It follows that \( x - y = 0. \) \( \square \)

**Remark 6.6.** Note that \( \{x_i + v\}_{i=1}^\infty \) is not Bessel. But we can scale it to be Bessel and it still does phase retrieval.
Corollary 6.7. We can perturb a family doing phase retrieval in $\ell_2$ as long as we perturb all vectors in the same direction.

Corollary 6.8. Given $\{x_i\}_{i=1}^{\infty}$ which is Bessel and does phase retrieval in $\ell_2$, for any $x_j$ the family $\{x_i - x_j\}_{i=1}^{\infty}$ does phase retrieval.

Proof. Let $v = -x_j$ and note that it follows that $\{x_i + v\}_{i=1}^{\infty}$ does phase retrieval by Theorem 6.5.

Note repeating the argument in the previous corollary, it is possible to “delete” a finite number of vectors by translating the system and scaling the set so they are Bessel.

Proposition 6.9. There is a family of vectors in $\ell_2$ doing phase retrieval where each of the vectors has all non-zero coordinates with respect to the unit vectors.

Proof. Let $\{x_i\}_{i=1}^{\infty}$ do phase retrieval. Let $\{e_i\}_{i=1}^{\infty}$ be the unit vectors. For any $j = 1, 2, \ldots$ the family $\{x_i(j)\}_{i=1}^{\infty}$ is a countable set of real numbers so choose a real number $a_j \neq -w_i(j)$ and $0 < a_j < \frac{1}{2^j}$ for all $i = 1, 2, \ldots$. Let $v = (a_1, a_2, \ldots)$. Then $\{x_i + v\}_{i=1}^{\infty}$ does phase retrieval and each vector has all non-zero coordinates.

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References
