Brownian motion under the influence of green noise

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Abstract

The motion of a Brownian particle is examined when exposed to green noise generated with a randomly localized potential function. To investigate this motion, an averaging method was developed which is valid for any intensity external noises. For the case of a limited quasi-periodic model potential, the particle trajectory implementations were numerically calculated. It is shown that introducing the effective potential in a certain manner plays a critical role when studying the possibility of a phase transition in a given system.

1. Introduction

If the spectral density of a random process is not constant as a function of frequency, then this noise is referred to as colored noise as opposed to white noise. Ornstein–Uhlenbeck process[1,2], band-limited white noise[3], dichotomies noise (or random-telegraph-signal)[4--6] and other noises are colored noises. For all these noises, the spectral density disappears at high frequencies. If color terminology is used, then this kind of noise should be called "red" noise. In our works[7,8] we examined a new kind of colored noise, the noise with vanishing density at zero frequency. This noise under certain conditions was called "green" noise. (There are works where the intermediate case between red and green noise is viewed[9,10] .) Let's review this issue in detail.

2. Green noise

Let's examine an RC-chain connected to a white noise generator $w(t)$, as shown in Fig. 1.

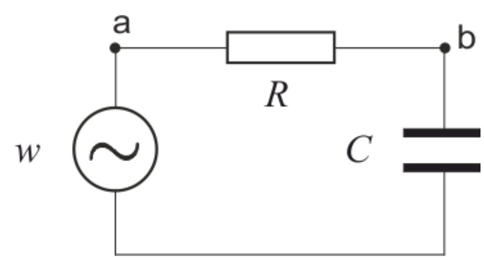

Fig. 1. White noise generator *w(t)* loading a RC chain.

As it results from Kirchhoff's Law,

$$RI + \frac{q}{C} = w(t),$$

where $q(t)$ is the capacitor charge and $I = \dot{q}$ is the loop current. The resistor voltage $U_{ab} = \zeta_G$ is in line with equations

$$\zeta_G = RI = R\dot{q} = \dot{\xi}$$

$$R\dot{q} + \frac{Rq}{RC} = R\dot{q} + \gamma Rq = w,$$

where $\xi = Rq$ and $\gamma = 1/RC$, or

$$\zeta_G = \dot{\xi}, \qquad (1)$$

$$\dot{\xi} + \gamma\xi = w. \qquad (2)$$

The random process *ξ(t)* defined with equation (2) is called Ornstein–Uhlenbeck process. Considering that the correlation function of Gaussian white noise is $\Psi_w(\tau)=\langle w(t)w(t+\tau)\rangle = D_w\delta(\tau)$ from (2) results that process $\xi(t)$ correlation function is

$$\Psi_\xi(\tau)=\langle \xi(t)\xi(t+\tau)\rangle = \frac{D_w}{2\gamma}\exp\left(-\gamma\,|\,\tau\,|\right). \tag{3}$$

According to Wiener-Khinchin theorem, the spectral density of this process looks like a Lorentzian function

$$S_\xi(\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\Psi_\xi\,(\tau)\cos(\omega\tau)d\tau = \frac{1}{2\pi}\frac{D_w}{\omega^2+\gamma^2}.$$

Finally, from equation (1) results that process $\zeta_G(t)$ spectral density equals

$$S_G(\omega)=\frac{D_w}{2\pi}\frac{\omega^2}{\omega^2+\gamma^2}. \tag{4}$$

Accordingly,

$$\Psi_G(\tau)=\int_{-\infty}^{\infty} S_G(\omega)\cos(\omega\tau)d\omega = \frac{D_w}{2\pi}\int_{-\infty}^{\infty}\frac{\omega^2\cos(\omega\tau)d\omega}{\omega^2+\gamma^2} = D_w\left[\delta(\tau)-\frac{\gamma}{2}\exp(-\gamma\,|\,\tau\,|)\right]. \tag{5}$$

Let’s examine the integral of some stationary process

$$\rho(t)=\int_0^t \zeta(t')dt'.$$

Its dispersion equals

$$\sigma_\rho^2=\langle\rho^2(t)\rangle=\int_0^t\int_0^t \Psi_\zeta(t'-t'')dt'dt'' = D_\rho t + R(t),$$

where

$$D_\rho=\int_{-\infty}^{\infty}\Psi_\zeta(\tau)d\tau = 2\pi S_\zeta(0)$$

is the diffusion coefficient, and

$$R(t) = -2\int_0^\infty \tau \Psi_\zeta(\tau)d\tau + 2\int_t^\infty (\tau - t)\Psi_\zeta(\tau)d\tau .$$

This remainder term is limited as t increases, if the absolute value $\Psi_\zeta(\tau)$ decreases fast enough as $|\tau|$ increases at least as

$$|\Psi_\zeta(\tau)| = O(\tau^{-2}) . \qquad (6)$$

If $S(0) = 0$ and the condition (6) is complied with, then diffusion disappears and a noise class appears, which we conditionally will call *green* noise due to red color deficiency in the zero-frequency region. It follows from correlations (4) and (5) that the noise equalling the time derivative of the Ornstein-Uhlenbeck process is a green noise.

3. A Brownian Particle in a Fluctuating Potential

Next, we will study the Brownian motion of a charged particle relative to a molecule that creates a potential $V(y)$. Let's suppose that a molecule randomly moves relative to an inertial space and its coordinate time dependence is described with a random stationary process $\xi(t)$. In dimensionless variables, we will write the equation of a Brownian particle motion as

$$\ddot{y} = -\beta\dot{y} - \varepsilon\frac{\partial U\left[y + \xi(t)\right]}{\partial y}, \qquad (7)$$

where $y(t)$ is the particle normalized coordinate, β is inversely proportional to its mass and $U\left[y + \xi(t)\right] = F \cdot (y + \xi) + V(y + \xi)$ is a dimensionless potential. The quantity F is proportional to some constant force acting on the particle.

Let's introduce a new dynamic variable

$$x = y + \xi ,$$

which is the particle coordinate of the molecule frame of reference. Then, it follows from (7):

$$\ddot{x} = -\beta\dot{x} - \varepsilon\frac{\partial U(x)}{\partial x} + \ddot{\xi} + \beta\dot{\xi} . \qquad (8)$$

Green noise has appeared on the right side of this equation.

To solve the problem, we will use the averaging method, assuming ε to be a small parameter. We will seek a solution of this equation as

$$x(t) = \bar{y}(t) + x_0(t) + \varepsilon u(t, \bar{y}), \qquad (9)$$

where $\bar{y}$ is a slow motion, $x_0(t)$ is a zero approach, when ε = 0 and $u(t,\ \bar{y}\ )$ is some correction process. In our case, $x_0(t) = \xi(t)$. Assuming that $\xi(t)$ is a fast motion, the equation for $\bar{y}$ follows from Eq. (7) with averaging of the right-hand side at $\bar{y}$ fixed value, i.e.

$$\ddot{\bar{y}} = -\beta\dot{\bar{y}} - \varepsilon \frac{\partial U_{eff}(\bar{y})}{\partial \bar{y}} ,$$

where

$$U_{eff}(\bar{y}) = \lim_{T\to\infty} \int_t^{t+T} U(\bar{y} + \xi(t))dt = \langle U(\bar{y} + \xi(t)) \rangle .$$

Here, it is assumed that the random process $U(\bar{y} + \xi(t))$ with fixed $\bar{y}$ is ergodic.

We will substitute (9) into (8). We will find out that correction process $u(t, \bar{y})$ is in line with equation

$$\ddot{u} = -\beta\dot{u} - \frac{\partial U(\bar{y} + \xi + \varepsilon u)}{\partial \bar{y}} - \frac{\partial U_{eff}(\bar{y})}{\partial \bar{y}} .$$

Expanding the right-hand side of this equation into a Taylor series with respect to the parameter ε, we will obtain a system of equations for successive approximations of the averaging method. Note that the developed procedure is valid for any noise intensity $\xi(t)$, i.e. the noise need not be weak.

4. Special Case

Let's examine as an example the model potential consisting of the analytical expression

$$U(y) = -Fy - 0.5\left[1 + \cos\left(6\pi y\right)\right]\exp(-0.07 \cdot y^{20}) . \qquad (10)$$

This potential is shown in Fig. 2 as curve 1. For further calculations, we will assume that the process $\xi(t)$ is an Ornstein-Uhlenbeck process, i.e. a Gaussian process. Then,

$$U_{eff}(\bar{y}) = -F\bar{y} - \frac{1}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \{1 + \cos[6\pi(\bar{y} - z)]\} \exp\left[-0.07 \cdot (\bar{y} - z)^{20} - \frac{z^2}{2\sigma^2}\right] dz .$$

For $\sigma^2 = 2$ and $F = 0.1$, the numerical calculation is shown in Fig. 2 as curve 2. It is apparent that the fine structure available in the real potential has completely disappeared in the effective potential and it represents some potential well with a single barrier. σ being significantly larger than the period of the real potential oscillations is responsible for this.

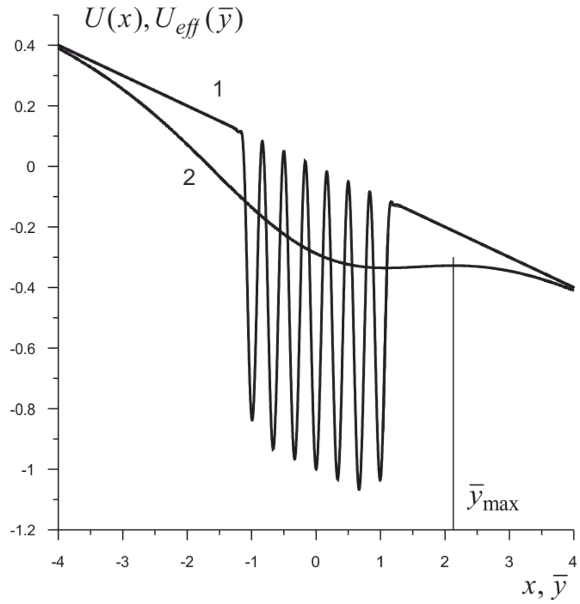

Fig. 2. Real and effective potentials.

Now, let's examine processes $x(t)$ and $y(t)$ implementation. Let's introduce new variables $x_1 = x$ $x_2 = \dot{y} = \dot{x} - \dot{\xi}$ and $x_3 = \xi$, allowing for transforming Eq. (8) into an equation system considering (2)

$$\dot{x}_1 = x_2 - \gamma x_3 + a\eta,$$

$$\dot{x}_2 = -\beta x_2 + \varepsilon f(x_1),$$

$$\dot{x}_3 = -\gamma x_3 + a\eta,$$

where $\eta(t)$ is white noise with single intensity, $\langle \eta(t)\eta(t+\tau)\rangle = \delta(\tau)$, $a = \sqrt{2\gamma\sigma^2}$, $\sigma^2 = D_w/2\gamma$ is Ornstein-Uhlenbeck process dispersion (see (3)), $f(x_1) = -\partial U(x_1)/\partial x_1$. Algorithms of numerical integration for this system of $O(h^{1/2})$ order are as[11].

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x1 += a*Z1+(x2-g*x3)*h;

x2 += (-b*x2+eps*f)*h;

x3 += a*Z1-g*x3*h;
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Here, $Z_1 = X\sqrt{h}$ is a random normally distributed number with single dispersion[12], h is an integration step, $g=\gamma$ and $b=\beta$. The initial conditions are chosen to be zero.

Fig. 3 shows $x(t)$ process implementation with full number of steps $N=10^{10}$ and $h=10^{-6}$. It is seen that in $t \approx 2200$ region a phase transition from locking state into running state occurs[13].

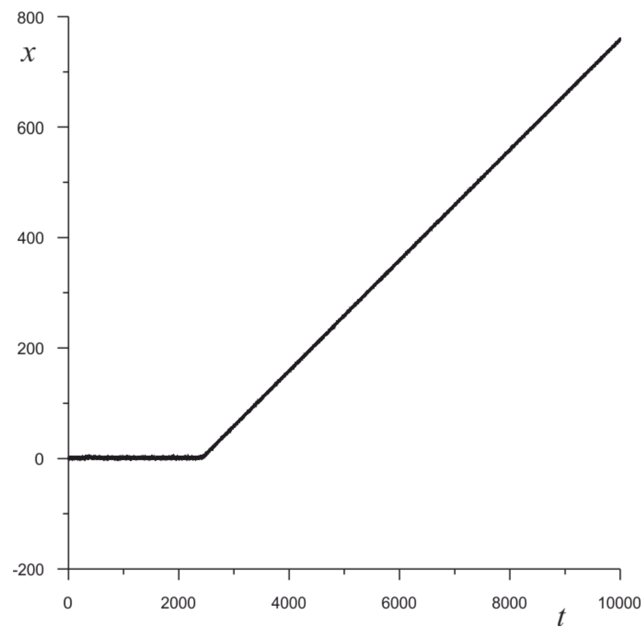

Fig 3. Implementation of process $x(t)=x_1(t)$. Here, $F=0.1$, $\beta=1$, $\gamma=10$, $\sigma^2=2$ and $\varepsilon=1$.

Let's examine this region in detail. Fig. 4 shows this region with high magnification. It is seen that the phase transition occurs when the slow motion $y(t)$ confidently crosses the barrier of effective potential $\bar{y}_{\max}$. If the barrier is too high for some values of parameters F, β, γ, σ, ε and the slow motion does not reach it, then phase transition does not occur.

For example, for $\sigma^2=1.5$ effective potential $U_{eff}(\bar{y})$ is shown in Fig. 5. Accordingly, slow motion implementation is shown in Fig. 6. It is seen that it does not reach the barrier at point $\bar{y}_{\max}$ and phase transition does not occur. At the same time, the process $y(t)$ is exposed to fluctuations around the minimum of the effective potential $\bar{y}_{\min}$ (possibly with a small offset due to random correction function $u(t,\bar{y})$.

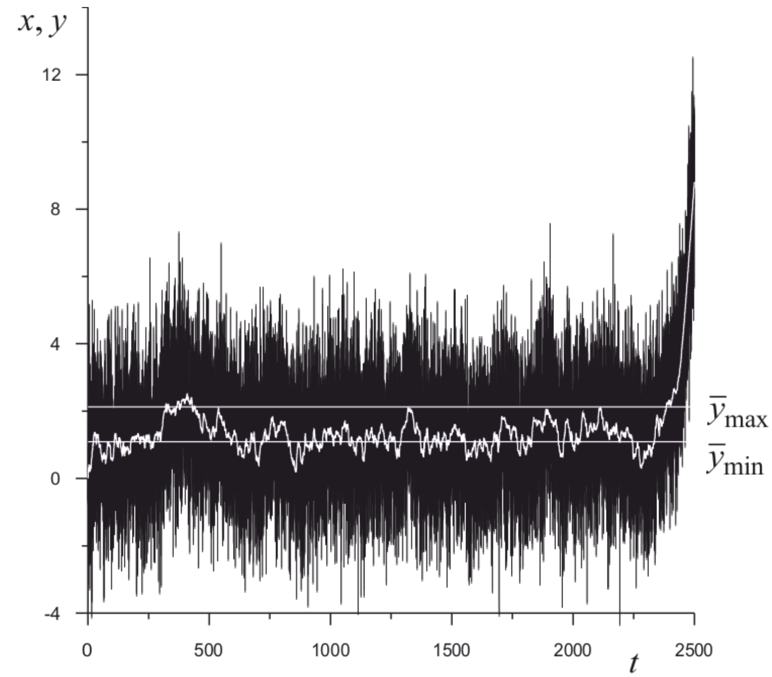

Fig. 4. The black noise curve is process $x(t)$, the white noise curve is process $y(t) = x(t) - \xi(t)$ - slow motion, straight lines $\overline{y}_{min}$ and $\overline{y}_{max}$ are effective potential minimum and maximum.

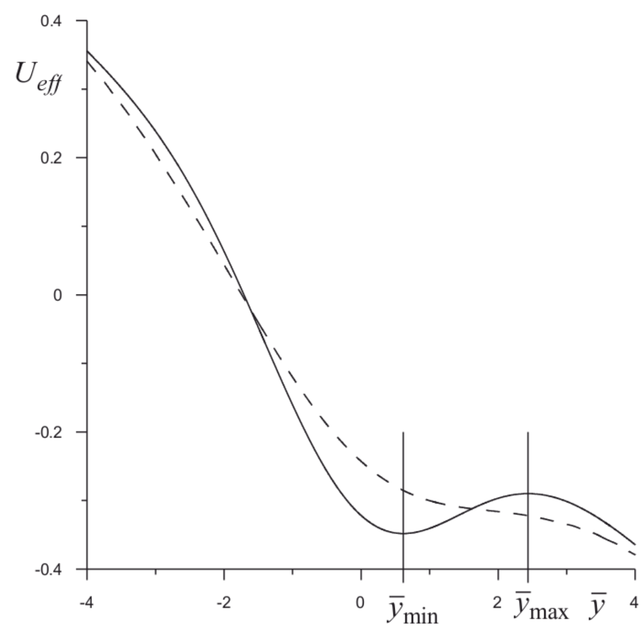

Fig. 5. Effective potential at $\sigma^2 = 1.5$ (continuous curve) and $\sigma^2 = 3$ (dashed curve).

The effective potential at $\sigma^2 = 3$ lacks maximum or minimum, and the process implementation represents simply a straight line $y \approx Ft$.

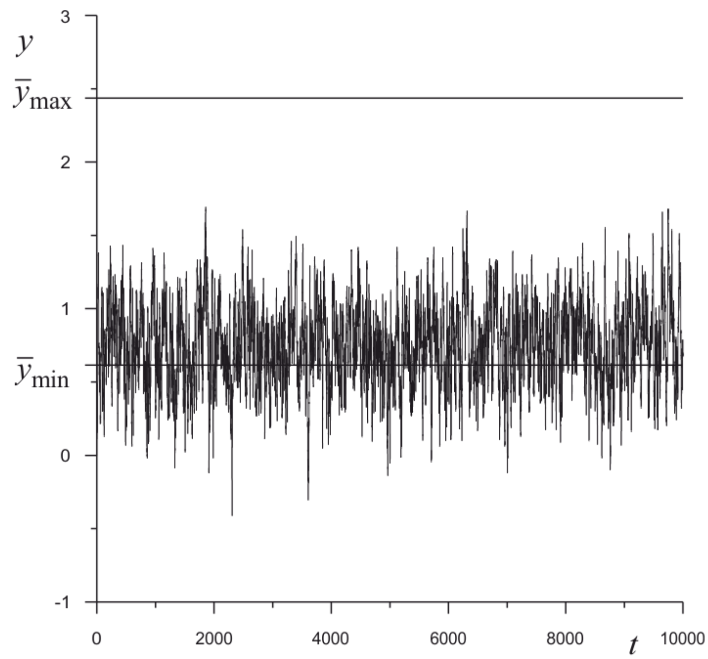

Fig. 6. Process *y(t)* implementation at $\sigma^2 = 1.5$.

Finally, it is important to note that the quasi-periodic potential of type (10) boundedness leads to a completely different description of the Brownian particle motion compared to that of a periodic potential[14]. As follows from this paper, this boundedness reduces to solving the problem of a particle motion in a solitary potential well.

5. Conclusion

Dynamics of a motion of a Brownian particle in the fluctuating potential under the influence of green noise is completely spotted by effective potential. Cross by a slow motion of a maximum of this potential leads to phase transition from a locking state to a running state of the continuous drift. If such cross is not present, the particle is in an locking state all the time. In the absence of extremes in effective potential, the particle moves almost as the free particle. All of this is well featured in the variant of a method of averaging developed here which is valid for noise with any intensity.

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